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Time Series Analysis Using Seasonal ARIMA Models

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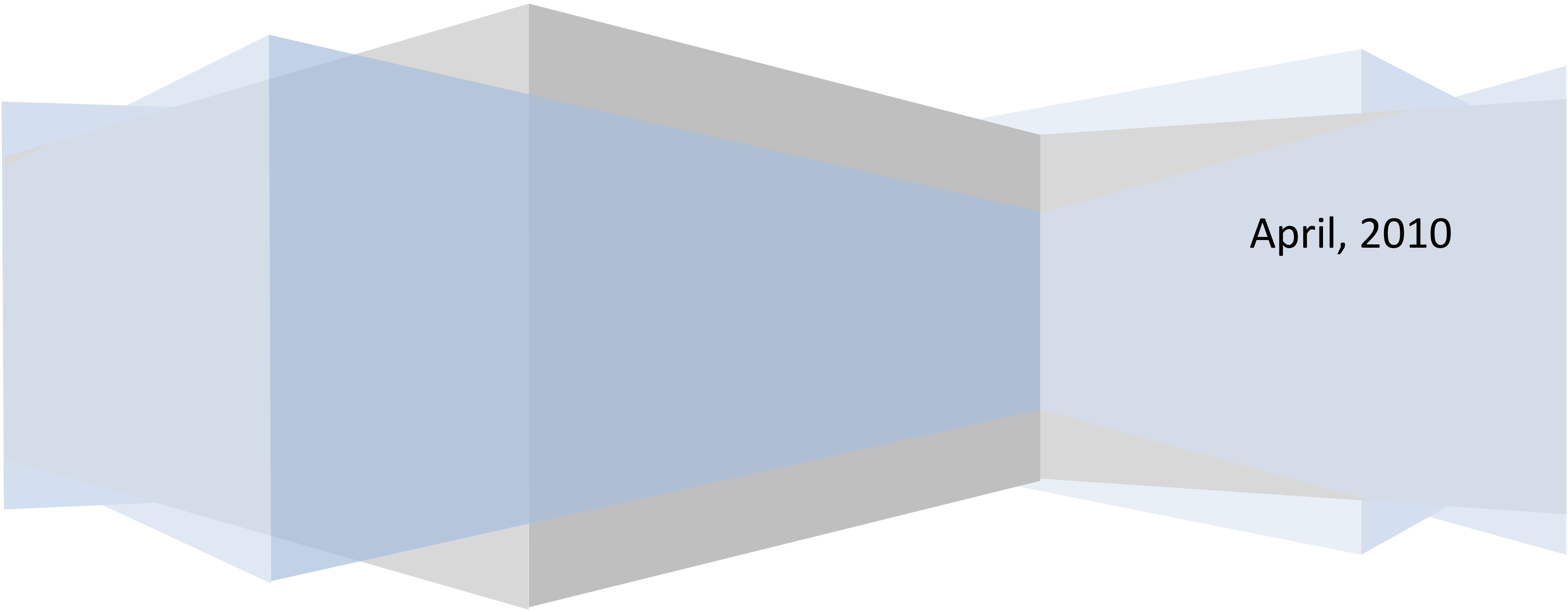

April, 2010

In this paper: 1) run a data analysis using Seasonal ARIMA models; 2) select the appropriate parameters for the Seasonal ARIMA model; 3) assess the fit and perform inference. Software for analyzing is SAS.

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In this paper: 1) run a data analysis using Seasonal ARIMA models; 2) select the appropriate parameters for the Seasonal ARIMA model; 3) assess the fit and perform inference. Software for analyzing is SAS.

1. Purpose

1) Run a data analysis using Seasonal ARIMA models;
2) Select the appropriate parameters for the Seasonal ARIMA model;
3) Assess the fit and perform inference.

2. Data description

A set of 132 time series data collected monthly was given as below:

14	18	23	36	39	39	40	40	40	39
28	13	18	21	23	37	42	42	42	44
45	43	33	14	15	17	26	33	54	53
51	54	54	52	37	18	21	26	34	49
64	65	63	61	60	57	42	23	23	29
35	47	57	57	56	55	53	52	39	25
26	27	36	53	63	65	65	64	64	62
48	27	26	26	35	57	61	62	52	62
62	60	52	33	32	31	43	71	79	80
79	75	73	72	59	34	34	32	48	77
92	94	95	97	97	93	83	43	35	43
63	97	133	134	133	134	134	133	121	58
58	51	80	127	150	151	150	151	151	148
132	63								

This data set has more than 50 observations, which means the sample size is large enough to be applied in time series data analysis.

3. Model identification

3.1. Plot $\{Z_t\}$ and transform the data into stationary one if necessary

In this case, let data be expressed by $\{Z_t\}$.
First of all, we need to plot data and examine. That is, we use a visual inspection to determine whether or not the time series is stationary. If not, the series should be transformed.

1) Plot $\{Z_t\}$

The SAS code to plot $\{Z_t\}$ is as follows:

```
options ps=80 ls=78;     /* sets page length and column width */
data data1;              /* name of data set */
  input zt @@;           /* read horizontal data and store in variable zt */
  date=_n_;              /* use the case number as date */
  datalines;
   14   18   23   36   39   39   40   40   40   39
   28   13   18   21   23   37   42   42   42   44
   45   43   33   14   15   17   26   33   54   53
   51   54   54   52   37   18   21   26   34   49
   64   65   63   61   60   57   42   23   23   29
   35   47   57   57   56   55   53   52   39   25
   26   27   36   53   63   65   65   64   64   62
   48   27   26   26   35   57   61   62   52   62
   62   60   52   33   32   31   43   71   79   80
   79   75   73   72   59   34   34   32   48   77
   92   94   95   97   97   93   83   43   35   43
   63   97  133  134  133  134  134  133  121   58
   58   51   80  127  150  151  150  151  151  148
  132   63
;
proc gplot data=data1;     /* use gplot of scatter plot */
  plot zt*date;            /* plot zt versus date */
  symbol i=join v=dot;     /*  connect points in plot */
  run;
proc arima data=data1;       /*  use arima procedure */
  identify var=zt nlag=48;   /* produce sample acf and pacf*/
  estimate printall plot;
run;
quit;
```

We have the following output:

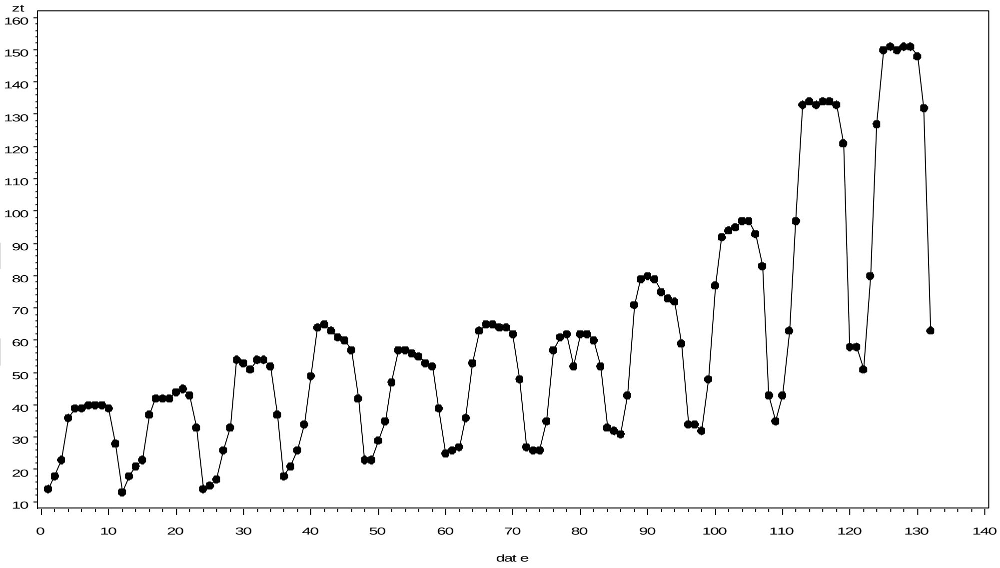

Figure 1 Raw time series data $\{Z_t\}$

```
                                 Autocorrelations
    Lag  Covariance  Correlation  -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
       0    1207.708      1.00000  |                    |********************|
       1    1096.771      0.90814  |                 .  |******************  |
       2     893.994      0.74024  |              .     |***************     |
       3     670.419      0.55512  |             .      |***********         |
       4     482.751      0.39973  |             .      |********            |
       5     363.958      0.30136  |            .       |****** .            |
       6     302.067      0.25012  |            .       |*****  .            |
       7     280.029      0.23187  |            .       |*****  .            |
       8     327.683      0.27133  |            .       |*****  .            |
       9     457.705      0.37899  |            .       |********            |
      10     630.870      0.52237  |            .       |**********          |
      11     782.786      0.64816  |           .        |*************       |
      12     849.805      0.70365  |           .        |**************      |
      13     761.560      0.63058  |          .         |*************       |
      14     593.060      0.49106  |          .         |**********          |
      15     404.637      0.33505  |         .          |*******   .         |
      16     245.146      0.20298  |         .          |****      .         |
      17     137.381      0.11375  |         .          |**        .         |
      18   75.512201      0.06253  |         .          |*         .         |
      19   50.512917      0.04183  |         .          |*         .         |
      20   84.519659      0.06998  |         .          |*         .         |
      21     185.035      0.15321  |         .          |***       .         |
      22     318.978      0.26412  |         .          |*****     .         |
      23     442.565      0.36645  |         .          |*******   .         |
      24     499.301      0.41343  |         .          |********  .         |
      25     438.196      0.36283  |         .          |*******   .         |
      26     309.012      0.25587  |         .          |*****     .         |
      27     166.343      0.13773  |         .          |***       .         |
      28   40.930347      0.03389  |         .          |*         .         |
      29  -35.893834      -.02972  |         .         *|          .         |
      30  -75.334578      -.06238  |         .         *|          .         |
      31  -88.172188      -.07301  |         .         *|          .         |
      32  -59.511348      -.04928  |         .         *|          .         |
      33   26.089403      0.02160  |         .          |          .         |
      34     138.185      0.11442  |         .          |**        .         |
      35     239.677      0.19846  |         .          |****      .         |
      36     288.779      0.23911  |         .          |*****     .         |
      37     245.475      0.20326  |        .           |****       .        |
      38     143.685      0.11897  |        .           |**         .        |
      39   29.476103      0.02441  |        .           |           .        |
      40  -67.644049      -.05601  |        .          *|           .        |
      41    -123.434      -.10221  |        .         **|           .        |
      42    -149.460      -.12376  |        .         **|           .        |
      43    -150.840      -.12490  |        .         **|           .        |
      44    -123.708      -.10243  |        .         **|           .        |
      45  -51.372046      -.04254  |        .          *|           .        |
      46   44.094924      0.03651  |        .           |*          .        |
      47     130.664      0.10819  |        .           |**         .        |
      48     171.625      0.14211  |        .           |***        .        |
```

Figure 2 ACF of the Original time series data $\{Z_t\}$

From the Figure 1 and Figure 2, we observe that:

- $\{Z_t\}$ indicates non-stationarity and has a kind of upward trend that means there are time-dependent means, and we can use (1-B) differencing to make it go to stationary;
- $\{Z_t\}$'s variances seems to be bigger as times go by, which means that there is non-stationarity caused by time-dependent variability. It implies *log* or *square* transformation.

- The seasonal nature of $\{Z_t\}$ looks like apparent: the numbers increase dramatically from January with peaks occurring in the middle of the year, and then decrease in the rest of the year. The phenomenon repeats itself every 12 months, and thus the seasonal period is 12. It implies $(1-B^{12})$ differencing to be used.
- The plot of ACF clearly indicates a non-stationary series since the autocorrelations diminish very slowly.

Thus, we need to do is to transform the data from non-stationary to stationary.

2) Transform $\{Z_t\}$ with *log*

Since the variances of $\{Z_t\}$ seems to be bigger as times go by, we can use *log[*Z_t*]* to reduce such non-stationarity.
The SAS code is below:

```
options ps=80 ls=78;   /* sets page length and column width */
data data1;            /* name of data set */
  input zt @@;         /* read horizontal data and store in variable zt */
  vt=log(zt);          /* transformation with log */
  date=_n_;            /* use the case number as date */
  datalines;
   14   18   23   36   39   39   40   40   40   39
   28   13   18   21   23   37   42   42   42   44
   45   43   33   14   15   17   26   33   54   53
   51   54   54   52   37   18   21   26   34   49
   64   65   63   61   60   57   42   23   23   29
   35   47   57   57   56   55   53   52   39   25
   26   27   36   53   63   65   65   64   64   62
   48   27   26   26   35   57   61   62   52   62
   62   60   52   33   32   31   43   71   79   80
   79   75   73   72   59   34   34   32   48   77
   92   94   95   97   97   93   83   43   35   43
   63   97   133  134  133  134  134  133  121  58
   58   51   80   127  150  151  150  151  151  148
   132  63
;
proc gplot data=data1;   /* use gplot of scatter plot */
  plot vt*date;          /* plot log(zt) versus date */
  symbol i=join v=dot;   /*  connect points in plot */
  run;
proc arima data=data1;      /*  use arima procedure */
  identify var=vt nlag=48;  /* produce sample acf and pacf */
  estimate printall plot;
run;
quit;
```

We have the following output of the plot of *{*V_t*}* where $V_t = log[Z_t]$:

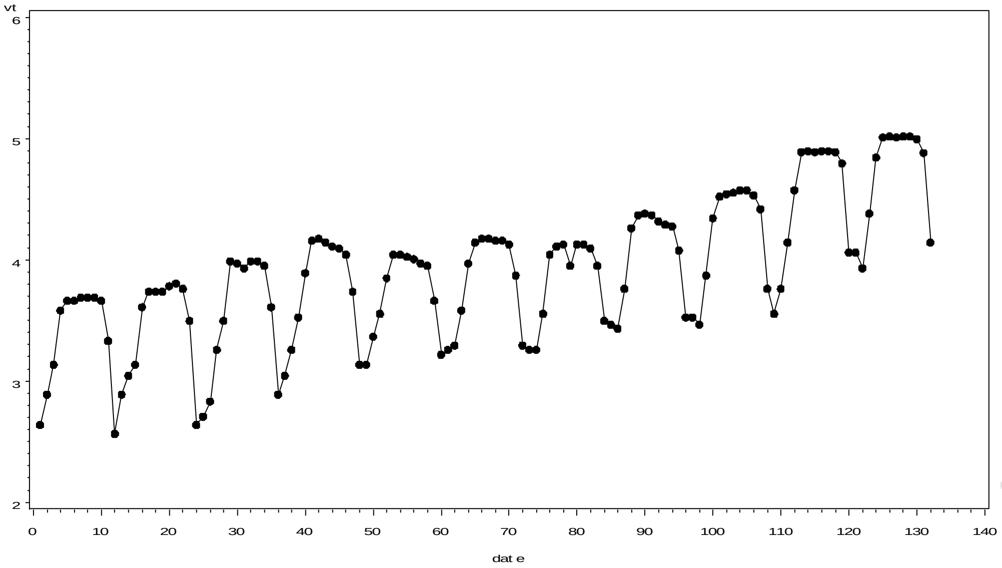

Figure 3 *log* transformation of raw data $\{Z_t\}$

From the Figure 3, we observe that: the non-stationarity caused by time-dependent variability is reduced by using *log*$[Z_t]$ transformation.

3) Transform $\{Z_t\}$ with *log* and *(1-B)* differencing

Since $\{Z_t\}$ has a kind of upward trend, we can use *(1-B)* differencing to reduce such non-stationarity. The SAS code is below:

```
options ps=80 ls=78;     /* sets page length and column width */
data data1;              /* name of data set */
  input zt @@;           /* read horizontal data and store in variable zt */
  vt=log(zt);            /* transformation with log */
  wt=dif(vt);            /* (1-B) differencing */
  date=_n_;              /* use the case number as date */
  datalines;
  14   18   23   36   39   39   40   40   40   39
  28   13   18   21   23   37   42   42   42   44
  45   43   33   14   15   17   26   33   54   53
  51   54   54   52   37   18   21   26   34   49
  64   65   63   61   60   57   42   23   23   29
  35   47   57   57   56   55   53   52   39   25
  26   27   36   53   63   65   65   64   64   62
  48   27   26   26   35   57   61   62   52   62
  62   60   52   33   32   31   43   71   79   80
  79   75   73   72   59   34   34   32   48   77
  92   94   95   97   97   93   83   43   35   43
  63   97   133  134  133  134  134  133  121  58
  58   51   80   127  150  151  150  151  151  148
  132  63
;
proc gplot data=data1;    /* use gplot of scatter plot */
  plot wt*date;           /* plot wt versus date */
  symbol i=join v=dot;    /* connect points in plot */
  run;
proc arima data=data1;       /* use arima procedure */
```

```
  identify var=wt nlag=48;   /* produce sample acf and pacf */
  estimate printall plot;
run;
quit;
```

We have the following output for the plot of $\{W_t\}$ where $W_t = (1-B)V_t = (1-B)\log[Z_t]$:

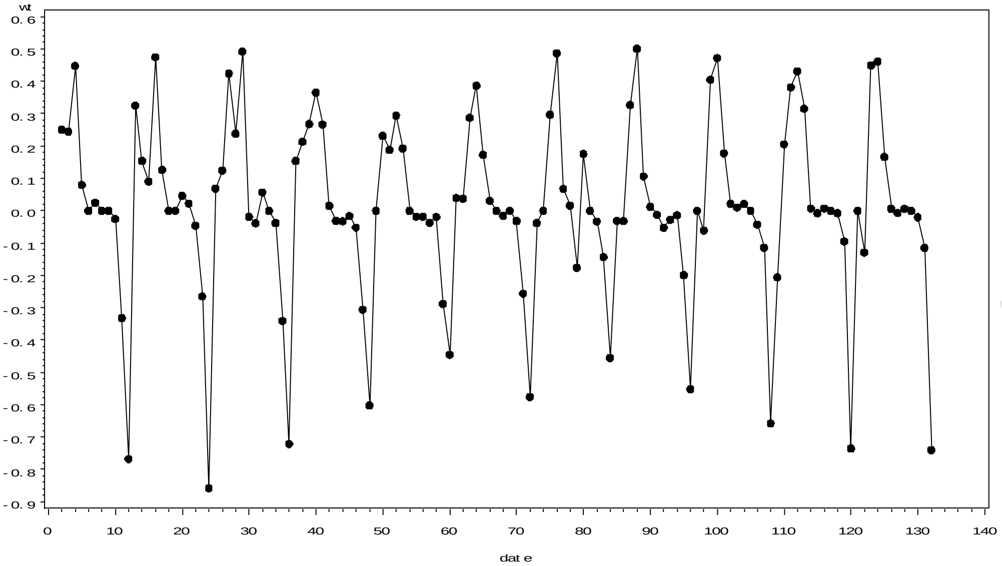

Figure 4 Transform $\{Z_t\}$ with *log* and *(1-B)* differencing

From the Figure 4, we observe that: the non-stationarity caused by time-dependent means for the upward trend is reduced by using *(1-B)* differencing.

4) Transform $\{Z_t\}$ with *log* and *(1-B)* differencing and $(1-B^{12})$

Since $\{Z_t\}$ has seasonal component with the period of 12 months, we can use $(1-B^{12})$ to reduce such non-stationarity.
The SAS code is below:

```
options ps=80 ls=78;     /* sets page length and column width */
data data1;              /* name of data set */
  input zt @@;           /* read horizontal data and store in variable zt */
  vt=log(zt);            /* transformation with log */
  wt=dif(vt);            /* (1-B) differencing */
  yt=dif12(wt);          /* (1-B)(1-B12) differening for log(zt) */
  date=_n_;              /* use the case number as date */
  datalines;
 14   18   23   36   39   39   40   40   40   39
 28   13   18   21   23   37   42   42   42   44
 45   43   33   14   15   17   26   33   54   53
 51   54   54   52   37   18   21   26   34   49
 64   65   63   61   60   57   42   23   23   29
 35   47   57   57   56   55   53   52   39   25
 26   27   36   53   63   65   65   64   64   62
 48   27   26   26   35   57   61   62   52   62
 62   60   52   33   32   31   43   71   79   80
 79   75   73   72   59   34   34   32   48   77
 92   94   95   97   97   93   83   43   35   43
```

```
   63    97    133   134   133   134   134   133   121   58
   58    51    80    127   150   151   150   151   151   148
   132   63
;
proc gplot data=data1;      /* use gplot of scatter plot */
  plot yt*date;             /* plot yt versus date */
  symbol i=join v=dot;      /* connect points in plot */
  run;
proc arima data=data1;      /* use arima procedure */
  identify var=yt nlag=48;  /* produce sample acf and pacf */
  estimate printall plot;
run;
quit;
```

We have the following output for the plot of {Y_t} where $Y_t = (1-B^{12})W_t = (1-B)\ (1-B^{12})\log[Z_t]$:

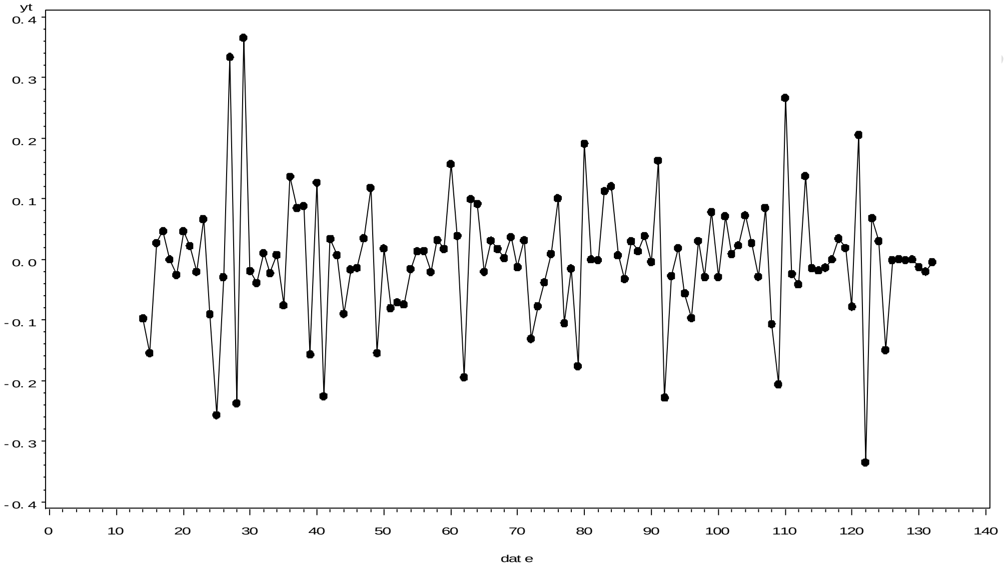

Figure 5 Transform {Z_t} with *log* and $(1-B)$ differencing and $(1-B^{12})$

```
                                            Autocorrelations
Lag    Covariance    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
  0      0.011014        1.00000    |                    |********************|
  1    -0.0037449        -.34000    |             *******|   .                |
  2    0.00013230        0.01201    |                .   |   .                |
  3     0.0011373        0.10326    |                .   |** .                |
  4    -0.0011032        -.10016    |                . **|   .                |
  5    -0.0008532        -.07746    |                . **|   .                |
  6    0.00039547        0.03590    |                .   |*  .                |
  7    -6.9465E-6        -.00063    |                .   |   .                |
  8    0.00002878        0.00261    |                .   |   .                |
  9    -0.0004626        -.04200    |                .  *|   .                |
 10    0.00042284        0.03839    |                .   |*  .                |
 11     0.0017290        0.15697    |                .   |***.                |
 12    -0.0041618        -.37785    |            ********|   .                |
 13     0.0018888        0.17149    |               .    |*** .               |
 14    -0.0007477        -.06788    |               .   *|    .               |
 15    -0.0017360        -.15761    |               . ***|    .               |
 16     0.0013723        0.12459    |               .    |**  .               |
 17    0.00042176        0.03829    |               .    |*   .               |
 18    -0.0012764        -.11589    |               .  **|    .               |
 19    0.00073314        0.06656    |               .    |*   .               |
```

```
20  0.00040871       0.03711   |               .    |*   .               |
21  0.00046728       0.04242   |               .    |*   .               |
22  -0.0009357       -.08495   |               .  **|    .               |
23  0.00017085       0.01551   |               .    |    .               |
24  0.00043193       0.03921   |               .    |*   .               |
25  -0.0008931       -.08109   |               .  **|    .               |
26  0.00009309       0.00845   |               .    |    .               |
27  0.00097275       0.08832   |               .    |**  .               |
28  0.00005150       0.00468   |               .    |    .               |
29  -0.0007789       -.07071   |               .   *|    .               |
30   0.0021393       0.19423   |               .    |****.               |
31  -0.0005161       -.04685   |               .   *|    .               |
32  -0.0006707       -.06090   |               .   *|    .               |
33  -0.0001356       -.01231   |               .    |    .               |
34  0.00058514       0.05312   |               .    |*   .               |
35  -0.0009827       -.08922   |               .  **|    .               |
36  -0.0000902       -.00819   |               .    |    .               |
37  0.00085806       0.07790   |               .    |**  .               |
38  0.00013833       0.01256   |               .    |    .               |
39  -0.0008577       -.07787   |               .  **|    .               |
40  0.00044116       0.04005   |               .    |*   .               |
41  -0.0000830       -.00753   |               .    |    .               |
42  -0.0012096       -.10982   |               .  **|    .               |
43  0.00071664       0.06506   |               .    |*   .               |
44  0.00032613       0.02961   |               .    |*   .               |
45  -0.0011925       -.10827   |               .  **|    .               |
46  -0.0000313       -.00284   |               .    |    .               |
47   0.0015668       0.14225   |               .    |*** .               |
48  -0.0007015       -.06369   |               .   *|    .               |
```

Figure 6 Plot of ACF of $Y_t = (1-B^{12})W_t = (1-B) (1-B^{12})log[Z_t]$

From the Figure 5, we observe that: the non-stationarity caused by seasonal component is reduced by using a seasonal differencing *$(1-B^{12})$*.
The ACF plot shows the autocorrelations diminishing fairly quickly, and only two significant spikes at lag 1 and lag 12.
Thus, here we can use the following transformation to conduct SARIMA analysis:

$$Y_t = (1-B) (1-B^{12})log[Z_t]$$

3.2. Check ACF and PACF to identify the model

According to the SAS code in 4) of 3.1., we also get the following plot of **PACF** for $\{Y_t\}$ where $Y_t = (1-B) (1-B^{12})log[Z_t]$:

```
                            Partial Autocorrelations

Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
  1       -0.34000    |             *******|   .                |
  2       -0.11713    |                . **|   .                |
  3        0.07795    |                .   |** .                |
  4       -0.03959    |                .  *|   .                |
  5       -0.13760    |                .***|   .                |
  6       -0.06144    |                .  *|   .                |
  7        0.00299    |                .   |   .                |
  8        0.02180    |                .   |   .                |
  9       -0.06293    |                .  *|   .                |
 10       -0.01665    |                .   |   .                |
 11        0.19687    |                .   |****                |
 12       -0.29671    |              ******|   .                |
 13       -0.07936    |                . **|   .                |
```

```
   14       -0.08807     |                  . **|    .                  |
   15       -0.15120     |                  .***|    .                  |
   16       -0.02099     |                  .   |    .                  |
   17        0.00689     |                  .   |    .                  |
   18       -0.10802     |                  . **|    .                  |
   19       -0.05691     |                  .  *|    .                  |
   20        0.01384     |                  .   |    .                  |
   21        0.07968     |                  .   |**  .                  |
   22       -0.07656     |                  . **|    .                  |
   23        0.01215     |                  .   |    .                  |
   24       -0.09202     |                  . **|    .                  |
   25       -0.02860     |                  .  *|    .                  |
   26       -0.07442     |                  .  *|    .                  |
   27       -0.07208     |                  .  *|    .                  |
   28        0.07921     |                  .   |**  .                  |
   29       -0.04930     |                  .  *|    .                  |
   30        0.09155     |                  .   |**  .                  |
   31        0.07959     |                  .   |**  .                  |
   32       -0.02834     |                  .  *|    .                  |
   33       -0.03972     |                  .  *|    .                  |
   34        0.02234     |                  .   |    .                  |
   35        0.01827     |                  .   |    .                  |
   36       -0.08371     |                  . **|    .                  |
   37       -0.00594     |                  .   |    .                  |
   38        0.06138     |                  .   |*   .                  |
   39       -0.05844     |                  .  *|    .                  |
   40       -0.00060     |                  .   |    .                  |
   41       -0.08751     |                  . **|    .                  |
   42       -0.00260     |                  .   |    .                  |
   43        0.07062     |                  .   |*   .                  |
   44        0.02414     |                  .   |    .                  |
   45       -0.08318     |                  . **|    .                  |
   46       -0.09270     |                  . **|    .                  |
   47        0.08371     |                  .   |**  .                  |
   48        0.01159     |                  .   |    .                  |
```

Figure 7 Plot of PACF of $Y_t = (1-B^{12})W_t = (1-B)\ (1-B^{12})\log[Z_t]$

The PACF plot in Figure shows the autocorrelations diminishing fairly quickly, and three significant spikes at lag 1, lag 11 (possible) and lag 12, while the ACF plot in Figure 6 shows the autocorrelations diminishing fairly quickly, and only two significant spikes at lag 1 and lag 12.
From Figure 6 and Figure 7, the plots of ACF and PACF for $\{Y_t\}$ where $Y_t = (1-B)\ (1-B^{12})\log[Z_t]$ before being modeled look like some kind of tail off after lag some times.
Thus, we start to try ARMA(1, 1) model first for $\{Y_t\}$, **that is ARIMA(1,1,1)×(1,1,1)$_{12}$ as a tentative model for the series *log{Z$_t$}***:

$$\Phi_1(B^{12})*\phi_1(B)*(1-B)\ (1-B^{12})\log[Z_t] = \theta_1(B)*\Theta_1(B^{12})a_t$$

i.e.,

$$(1-\Phi_1B^{12})*(1-\phi_1B)*(1-B)\ (1-B^{12})\log[Z_t] = (1-\theta_1B)*(1-\Theta_1B^{12})*a_t$$

Since the ACF plot in Figure 6 shows only two significant spikes at lag 1 (for non-seasonal component) and lag 12 (for seasonal component). This indicates the model may be just related to MA(1) and MA(1)$_{12}$. Thus, we use the **backward selection** process from the tentative model **ARIMA(1,1,1)×(1,1,1)$_{12}$**.

3.3. Check H_0: $\mu=0$ or H_0: $\theta_0=0$ when d>0

The 'Constant Estimate' is for the mean term μ or the intercept parameter.
We use the following SAS code:

```
proc arima data=data1;        /*  use arima procedure */
  identify var=yt nlag=48;    /* produce sample acf and pacf */
  estimate p=(1)(12) q=(1)(12) printall plot;
run;
```

The relevant output is:

```
                             The ARIMA Procedure
                     Conditional Least Squares Estimation
                                     Standard                 Approx
Parameter         Estimate            Error    t Value    Pr > |t|      Lag
MU              0.00007342        0.0023436       0.03      0.9751        0
MA1,1              0.74021          0.13938       5.31      <.0001        1
MA2,1              0.37838          0.20491       1.85      0.0674       12
AR1,1              0.43142          0.18674       2.31      0.0227        1
AR2,1             -0.10620          0.22424      -0.47      0.6367       12
```

Since the p-value for μ is 0.9751, we conclude that: we accept H_0: $\mu=0$ or H_0: $\theta_0=0$ when $d>0$ in this case.

4. Estimation and model selection

We start to try ARMA(1, 1) model first for *{Y_t}*, **that is ARIMA(1,1,1)×(1,1,1)$_{12}$** as a tentative model for *log{Z_t}*. Since the ACF plot in Figure 6 shows only two significant spikes at lag 1 (for non-seasonal component) and lag 12 (for seasonal component). This indicates the model may be just related to MA(1) and MA(1)$_{12}$. Thus, we use the **backward selection** procedure for the tentative model **ARIMA(1,1,1)×(1,1,1)$_{12}$** .

Here we will estimate four models: ARIMA(1,1,1)×(1,1,1)$_{12}$, ARIMA(1,1,1)×(0,1,1)$_{12}$, ARIMA(0,1,1)×(0,1,1)$_{12}$, and ARIMA(0,1,1)×(0,1,0)$_{12}$.

1) Try **ARIMA(1,1,1)×(1,1,1)$_{12}$** for *log{Z_t}*

```
options ps=80 ls=78;    /* sets page length and column width */
data data1;             /* name of data set */
  input zt @@;          /* read horizontal data and store in variable zt */
  vt=log(zt);           /* transformation with log */
  wt=dif(vt);           /* (1-B) differencing */
  yt=dif12(wt);         /* (1-B)(1-B12) differeing for log(zt) */
  date=_n_;             /* use the case number as date */
  datalines;
   14   18   23   36   39   39   40   40   40   39
   28   13   18   21   23   37   42   42   42   44
   45   43   33   14   15   17   26   33   54   53
   51   54   54   52   37   18   21   26   34   49
   64   65   63   61   60   57   42   23   23   29
   35   47   57   57   56   55   53   52   39   25
   26   27   36   53   63   65   65   64   64   62
   48   27   26   26   35   57   61   62   52   62
   62   60   52   33   32   31   43   71   79   80
   79   75   73   72   59   34   34   32   48   77
   92   94   95   97   97   93   83   43   35   43
   63   97  133  134  133  134  134  133  121   58
   58   51   80  127  150  151  150  151  151  148
  132   63
;
proc gplot data=data1;      /* use gplot of scatter plot */
```

```
  plot yt*date;             /* plot ytt versus date */
  symbol i=join v=dot;      /* connect points in plot */
  run;
proc arima data=data1;        /* use arima procedure */
  identify var=yt nlag=48;    /* produce sample acf and pacf */
  estimate p=(1)(12) q=(1)(12) noconstant printall plot;
run;
quit;
```

METHOD=CLS is the default. METHOD=CLS specifies the conditional least-squares method.

```
                      Conditional Least Squares Estimation

                                   Standard                 Approx
    Parameter      Estimate           Error    t Value    Pr > |t|     Lag

    MA1,1           0.73931         0.13911       5.31      <.0001       1
    MA2,1           0.37876         0.20375       1.86      0.0656      12
    AR1,1           0.43039         0.18623       2.31      0.0226       1
    AR2,1          -0.10561         0.22298      -0.47      0.6367      12

                          Variance Estimate      0.008405
                          Std Error Estimate     0.091676
                          AIC                    -227.061
                          SBC                    -215.944
                          Number of Residuals         119
                   * AIC and SBC do not include log determinant.
```

We need to take off ‘AR2,1’ since it is insignificant with a big p-value. We need further investigate ‘MA2,1’ since its p-value is close to 0.05.

2) Try **ARIMA(1,1,1)×(0,1,1)$_{12}$** for *log{Z_t}*

```
proc arima data=data1;        /* use arima procedure */
  identify var=yt nlag=48;    /* produce sample acf and pacf */
  estimate p=(1) q=(1)(12) noconstant printall plot;
run;
quit;
```

```
                      Conditional Least Squares Estimation

                                   Standard                 Approx
    Parameter      Estimate           Error    t Value    Pr > |t|     Lag

    MA1,1           0.70296         0.15226       4.62      <.0001       1
    MA2,1           0.46622         0.08471       5.50      <.0001      12
    AR1,1           0.38590         0.19746       1.95      0.0531       1

                          Variance Estimate      0.008342
                          Std Error Estimate     0.091337
                          AIC                    -228.912
                          SBC                    -220.575
                          Number of Residuals         119
                   * AIC and SBC do not include log determinant.
```

The information criteria AIC becomes more negative from *-227.061* to *-228.912*, and SBC becomes more negative from *-215.944* to *-220.575*. These imply significant improvement over the estimation.
We need to take off ‘AR1,1’ since they are insignificant with big p-value. Here, ‘MA2,1’ becomes significant with a very small p-value.

3) Try **ARIMA(0,1,1)×(0,1,1)$_{12}$** for *log{Z$_t$}*

```
proc arima data=data1;        /* use arima procedure */
  identify var=yt nlag=48;   /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
run;
```

```
                Conditional Least Squares Estimation
                             Standard                 Approx
Parameter      Estimate        Error    t Value     Pr > |t|     Lag
MA1,1           0.34986      0.08691       4.03       0.0001       1
MA2,1           0.43353      0.08560       5.06       <.0001      12


                   Variance Estimate      0.008304
                   Std Error Estimate     0.091125
                   AIC                    -230.444
                   SBC                    -224.886
                   Number of Residuals         119
            * AIC and SBC do not include log determinant.
```

The information criteria AIC becomes more negative from *-228.912* to *-230.444*, and SBC becomes **more** negative from *-220.575* to *-224.886*. These imply significant improvement over the estimation. 'MA1.1' and 'MA2,1' both are significant with very small p-values.

4) Try **ARIMA(0,1,1)×(0,1,0)$_{12}$** for *log{Z$_t$}*

```
proc arima data=data1;        /* use arima procedure */
    identify var=yt nlag=60;  /* produce sample acf and pacf */
    estimate q=(1) noconstant printall plot;
run;
quit;
```

```
                Conditional Least Squares Estimation
                             Standard                 Approx
Parameter      Estimate        Error    t Value     Pr > |t|     Lag
MA1,1           0.35923      0.08590       4.18       <.0001       1

                   Variance Estimate       0.00975
                   Std Error Estimate      0.09874
                   AIC                    -212.329
                   SBC                     -209.55
                   Number of Residuals         119
            * AIC and SBC do not include log determinant.
```

Here we observe that the estimation is **not good** since the information criteria AIC becomes **less** negative from *-230.444* to *-212.329*, and SBC becomes more negative from *-224.886* to *-209.55*.

5) Check other models approximating to **ARIMA(0,1,1)×(0,1,1)$_{12}$** for *{Z$_t$}*
Using the similar way to investigate p-values for coefficients and compare the information criteria AIC and SBC, we find that **ARIMA(0,1,1)×(0,1,1)$_{12}$** perform best.

Thus, steps of 1) ~ 5) imply we can select **ARIMA(0,1,1)×(0,1,1)$_{12}$** model for *{Z$_t$}*.

6) Check H_0: $\mu=0$ or H_0: $\theta_0=0$ when d>0
The 'Constant Estimate' is for the mean term μ or the intercept parameter.
We use the following SAS code:

```
proc arima data=data1;        /* use arima procedure */
  identify var=yt nlag=48;    /* produce sample acf and pacf */
  estimate q=(1)(12) printall plot;
run;
```

The output shows that:

```
                              The ARIMA Procedure
                        Conditional Least Squares Estimation

                                    Standard                 Approx
     Parameter        Estimate         Error    t Value    Pr > |t|      Lag
     MU             -0.0000902     0.0033259      -0.03      0.9784        0
     MA1,1             0.34984       0.08726       4.01      0.0001        1
     MA2,1             0.43348       0.08594       5.04      <.0001       12

                       Constant Estimate      -0.00009
                       Variance Estimate      0.008375
                       Std Error Estimate     0.091517
                       AIC                    -228.445
                       SBC                    -220.107
                       Number of Residuals         119
                 * AIC and SBC do not include log determinant
```

Since the p-value for μ is 0.9784, and the values of AIC and SBC become less negative, we conclude that: we accept H_0: μ=0 or H_0: θ_0=0 when d>0 in this case.

According to the results of the above, we conclude that:

- Therefore, we chose **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}*** in this case.

5. Diagnostic testing for residuals

If the fitted model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}*** is adequate, the residuals should be approximately white noise. So, we should check if the residuals have zero mean with constant variance and whether or not they are uncorrelated.
For the estimated model, the residuals $\hat{a}_t$'s are estimates of these unobserved white noise a_t's. If the model is appropriate most of the coefficients of the sample ACF and PACF should be close to zero.
We use the following SAS code:

```
proc arima data=data1;        /*  use arima procedure */
  identify var=yt nlag=60;    /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
run;
quit;
```

1) Check the residual plots

We obtain the plots of residuals for this model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}***:

```
                    Autocorrelation Plot of Residuals
Lag  Covariance  Correlation  -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
  0   0.0083038      1.00000  |                    |********************|
  1  0.00009288      0.01119  |                .   |   .                |
  2  -0.0001558      -.01876  |                .   |   .                |
  3  0.00010415      0.01254  |                .   |   .                |
  4  -0.0007902      -.09516  |                . **|   .                |
  5  -0.0010535      -.12687  |                .***|   .                |
  6  -0.0001567      -.01887  |                .   |   .                |
  7  -0.0002055      -.02474  |                .   |   .                |
  8  -0.0002735      -.03294  |                .  *|   .                |
  9  -0.0002564      -.03087  |                .  *|   .                |
 10  0.00014323      0.01725  |                .   |   .                |
 11  0.00063677      0.07668  |                .   |** .                |
 12  -0.0000721      -.00868  |                .   |   .                |
 13  0.00022088      0.02660  |                .   |*  .                |
 14  -0.0012106      -.14579  |                .***|   .                |
 15  -0.0012667      -.15255  |                .***|   .                |
 16  0.00068110      0.08202  |                .   |** .                |
 17  -0.0000918      -.01105  |                .   |   .                |
 18  -0.0002620      -.03156  |                .  *|   .                |
 19  0.00056078      0.06753  |                .   |*  .                |
 20  0.00022719      0.02736  |                .   |*  .                |
 21  0.00013887      0.01672  |                .   |   .                |
 22  -0.0006722      -.08095  |                . **|   .                |
 23  0.00013426      0.01617  |                .   |   .                |
 24  0.00026635      0.03208  |                .   |*  .                |
 25  -0.0004397      -.05296  |                .  *|   .                |
 26  -0.0004531      -.05456  |                .  *|   .                |
 27  0.00042783      0.05152  |                .   |*  .                |
 28  0.00060441      0.07279  |                .   |*  .                |
 29  -0.0000513      -.00618  |                .   |   .                |
 30   0.0014498      0.17459  |                .   |***.                |
 31  0.00039447      0.04751  |                .   |*  .                |
 32  -0.0006846      -.08244  |                . **|   .                |
 33  -0.0005450      -.06563  |                .  *|   .                |
 34  -0.0000759      -.00914  |                .   |   .                |
 35  -0.0002934      -.03533  |                .  *|   .                |
 36  -0.0001162      -.01399  |                .   |   .                |
 37  0.00051135      0.06158  |                .   |*  .                |
 38  -0.0003377      -.04067  |                .  *|   .                |
 39  -0.0003279      -.03949  |                .  *|   .                |
 40  0.00024254      0.02921  |                .   |*  .                |
 41  -7.9037E-6      -.00095  |                .   |   .                |
 42  -0.0003426      -.04126  |                .  *|   .                |
 43  0.00065336      0.07868  |                .   |** .                |
 44  -0.0004424      -.05328  |                .  *|   .                |
 45  -0.0011499      -.13848  |                .***|   .                |
 46  -0.0000166      -.00201  |                .   |   .                |
 47   0.0010148      0.12221  |                .   |** .                |
 48  -0.0003056      -.03680  |                .  *|   .                |
 49  -0.0000910      -.01095  |                .   |   .                |
 50  -0.0006832      -.08227  |                . **|   .                |
 51  0.00040066      0.04825  |                .   |*  .                |
 52  -0.0001154      -.01389  |                .   |   .                |
 53   0.0010975      0.13217  |                .   |***.                |
 54  0.00067642      0.08146  |                .   |** .                |
 55  0.00036040      0.04340  |                .   |*  .                |
 56  -0.0006161      -.07419  |                .  *|   .                |
 57  -0.0001564      -.01883  |               .    |    .               |
 58  -0.0001865      -.02246  |               .    |    .               |
 59  -0.0001443      -.01737  |               .    |    .               |
 60  -0.0000335      -.00403  |               .    |    .               |
```

Figure 8 ACF plot of residuals

```
                    Partial Autocorrelations
Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
 1        0.01119     |                .   |   .                |
 2       -0.01889     |                .   |   .                |
 3        0.01297     |                .   |   .                |
 4       -0.09587     |                . **|   .                |
 5       -0.12527     |                .***|   .                |
 6       -0.02164     |                .   |   .                |
 7       -0.02799     |                .  *|   .                |
 8       -0.04090     |                .  *|   .                |
 9       -0.05735     |                .  *|   .                |
10       -0.00442     |                .   |   .                |
11        0.06680     |                .   |*  .                |
12       -0.02228     |                .   |   .                |
13        0.01108     |                .   |   .                |
14       -0.16502     |                .***|   .                |
15       -0.14903     |                .***|   .                |
16        0.08918     |                .   |** .                |
17       -0.01440     |                .   |   .                |
18       -0.05474     |                .  *|   .                |
19       -0.00009     |                .   |   .                |
20        0.00427     |                .   |   .                |
21        0.02875     |                .   |*  .                |
22       -0.12256     |                . **|   .                |
23       -0.01402     |                .   |   .                |
24        0.02942     |                .   |*  .                |
25       -0.01828     |                .   |   .                |
26       -0.05876     |                .  *|   .                |
27        0.01091     |                .   |   .                |
28        0.07621     |                .   |** .                |
29       -0.04989     |                .  *|   .                |
30        0.16554     |                .   |***.                |
31        0.06110     |                .   |*  .                |
32       -0.10202     |                . **|   .                |
33       -0.04444     |                .  *|   .                |
34        0.03888     |                .   |*  .                |
35        0.02716     |                .   |*  .                |
36       -0.03781     |                .  *|   .                |
37        0.03237     |                .   |*  .                |
38       -0.00363     |                .   |   .                |
39       -0.04460     |                .  *|   .                |
40       -0.01652     |                .   |   .                |
41       -0.02255     |                .   |   .                |
42        0.00119     |                .   |   .                |
43        0.07946     |                .   |** .                |
44       -0.04678     |                .  *|   .                |
45       -0.07474     |                .  *|   .                |
46       -0.05167     |                .  *|   .                |
47        0.09133     |                .   |** .                |
48       -0.02364     |                .   |   .                |
49       -0.04436     |                .  *|   .                |
50       -0.15083     |                .***|   .                |
51        0.06298     |                .   |*  .                |
52        0.06340     |                .   |*  .                |
53        0.11746     |                .   |** .                |
54        0.00574     |                .   |   .                |
55        0.03781     |                .   |*  .                |
56       -0.05192     |                .  *|   .                |
57        0.02232     |                .   |   .                |
58       -0.03875     |                .  *|   .                |
59       -0.05251     |                .  *|   .                |
60       -0.03401     |                .  *|   .                |
```

Figure 9 PACF plot of residuals

We find that the sample ACF and sample PACF of the residuals do not form any pattern and all of them are (except $\rho_{\hat{\alpha}}(0)$=1) fall within the non-significance bounds where α=0.05. It implies the residuals are approximately white noise.

2) Portmanteau test for un-correlation among $\hat{a}_t$'s

H_0: $\rho_{\hat{\alpha}}(1) = \rho_{\hat{\alpha}}(2) = \ldots = \rho_{\hat{\alpha}}(K) = 0$
H_1: not H_0

The degrees of freedom of this statistic take into account the number of estimated parameters so the statistic test under H_0 follows approximately a $\chi^2_{(K-p-q)}$ distribution. The null hypothesis test is that the current set of autocorrelations is white noise.
The relevant SAS output is below:

```
                          Autocorrelation Check of Residuals
 To       Chi-          Pr >
Lag      Square    DF   ChiSq   ---------------Autocorrelations---------------
  6        3.29     4  0.5105    0.011   -0.019    0.013   -0.095   -0.127   -0.019
 12        4.47    10  0.9238   -0.025   -0.033   -0.031    0.017    0.077   -0.009
 18       11.80    16  0.7576    0.027   -0.146   -0.153    0.082   -0.011   -0.032
 24       13.78    22  0.9093    0.068    0.027    0.017   -0.081    0.016    0.032
 30       20.86    28  0.8313   -0.053   -0.055    0.052    0.073   -0.006    0.175
 36       23.33    34  0.9158    0.048   -0.082   -0.066   -0.009   -0.035   -0.014
 42       25.05    40  0.9689    0.062   -0.041   -0.039    0.029   -0.001   -0.041
 48       33.76    46  0.9098    0.079   -0.053   -0.138   -0.002    0.122   -0.037
 54       41.01    52  0.8638   -0.011   -0.082    0.048   -0.014    0.132    0.081
 60       42.97    58  0.9298    0.043   -0.074   -0.019   -0.022   -0.017   -0.004
```

We observe from the above that: all values of χ^2 are not big enough and p-value's are larger than 0.05. Then the H_0 of uncorrelated errors is not rejected.
This means that the residuals can be taken as white noise, and so the **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z$_t$}*** is an **adequate** model for this time series.

6. Final model selected

According to the above analysis, we conclude that:

- In this case, we chose **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z$_t$}***.

Use the following SAS code:

```
proc arima data=data1;        /*  use arima procedure */
  identify var=yt nlag=60;    /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
run;
```

We get:

```
                          The ARIMA Procedure
                   Conditional Least Squares Estimation

                          Standard                 Approx
Parameter         Estimate          Error    t Value    Pr > |t|      Lag
MA1,1              0.34986        0.08691       4.03      0.0001        1
MA2,1              0.43353        0.08560       5.06      <.0001       12
```

```
                  Variance Estimate         0.008304
                  Std Error Estimate        0.091125
                  AIC                       -230.444
                  SBC                       -224.886
                  Number of Residuals            119
           * AIC and SBC do not include log determinant.
```

Here, MA1,1=0.34986 with p-value=0.0001 and MA2,1=0.43353 with p-value=<.0001, and both of them are significant. Meanwhile the values of AIC and SBC are minimal.
Therefore, the final model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z$_t$}*** can be written as:

$$(1-B)\,(1-B^{12})\,\log\{Z_t\} = (1-0.34986*B)\,(1-0.43353*B^{12})a_t$$

where $\hat{\sigma}_a^2$=0.008304.

7. Forecasting

This final adequate model is selected as **ARIMA(1,1,1)×(1,1,1)$_{12}$ without constant for *log{Z$_t$}*** where $\hat{\sigma}_a^2$=0.008304.

7.1. Forecast check for Y_t

In the forecasting stage, we use the FORECAST statement in PROC ARIMA to forecast future values of Y_t, and it generates confidence intervals for these forecasts from the ARIMA model

$$Y_t = (1-0.34986*B)\,(1-0.43353*B^{12})a_t$$

where $Y_t = (1-B)\,(1-B^{12})\log\{Z_t\}$ and $\hat{\sigma}_a^2$=0.008304.

The following two SAS codes and relevant outputs are used to forecast 24 periods beginning with the end of the input series BACK=12 (lag 12) or BACK=0.

```
proc arima data=data1;        /*  use arima procedure */
  identify var=yt nlag=60;  /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
  forecast lead=24 back=12;
run;
```

```
                                Forecasts for variable yt
         Obs     Forecast Std Error   95% Confidence Limits       Actual     Residual
         121       0.1246    0.0911     -0.0540      0.3032       0.2059       0.0812
         122      -0.1035    0.0965     -0.2927      0.0857      -0.3345      -0.2310
         123      -0.0082    0.0965     -0.1974      0.1811       0.0683       0.0764
         124       0.0175    0.0965     -0.1717      0.2067       0.0306       0.0131
         125      -0.0716    0.0965     -0.2608      0.1177      -0.1492      -0.0776
         126       0.0048    0.0965     -0.1844      0.1940      -0.0008      -0.0056
         127      -0.0039    0.0965     -0.1931      0.1853       0.0008       0.0048
         128       0.0040    0.0965     -0.1852      0.1932      -0.0008      -0.0048
         129      -0.0032    0.0965     -0.1925      0.1860       0.0000       0.0032
         130      -0.0112    0.0965     -0.2004      0.1780      -0.0126      -0.0014
         131      -0.0244    0.0965     -0.2137      0.1648      -0.0199       0.0046
         132       0.0578    0.0965     -0.1314      0.2470      -0.0043      -0.0621
         133      -0.0167    0.1043     -0.2211      0.1877        .            .
         134       0.0000    0.1052     -0.2062      0.2062        .            .
         135       0.0000    0.1052     -0.2062      0.2062        .            .
         136       0.0000    0.1052     -0.2062      0.2062        .            .
         137       0.0000    0.1052     -0.2062      0.2062        .            .
```

```
     138      0.0000     0.1052     -0.2062      0.2062        .          .
     139      0.0000     0.1052     -0.2062      0.2062        .          .
     140      0.0000     0.1052     -0.2062      0.2062        .          .
     141      0.0000     0.1052     -0.2062      0.2062        .          .
     142      0.0000     0.1052     -0.2062      0.2062        .          .
     143      0.0000     0.1052     -0.2062      0.2062        .          .
     144      0.0000     0.1052     -0.2062      0.2062        .          .
```

```
proc arima data=data1;        /*  use arima procedure */
  identify var=yt nlag=60;    /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
  forecast lead=24 back=0;
run;
```

```
                          Forecasts for variable yt
  Obs         Forecast    Std Error        95% Confidence  Limits
  133         -0.0307       0.0911         -0.2093         0.1479
  134          0.1001       0.0965         -0.0891         0.2894
  135         -0.0331       0.0965         -0.2224         0.1561
  136         -0.0057       0.0965         -0.1949         0.1835
  137          0.0337       0.0965         -0.1556         0.2229
  138          0.0024       0.0965         -0.1868         0.1917
  139         -0.0021       0.0965         -0.1913         0.1871
  140          0.0021       0.0965         -0.1871         0.1913
  141         -0.0014       0.0965         -0.1906         0.1878
  142          0.0006       0.0965         -0.1886         0.1898
  143         -0.0020       0.0965         -0.1912         0.1872
  144          0.0269       0.0965         -0.1623         0.2161
  145         -0.0092       0.1043         -0.2136         0.1952
  146          0.0000       0.1052         -0.2062         0.2062
  147          0.0000       0.1052         -0.2062         0.2062
  148          0.0000       0.1052         -0.2062         0.2062
  149          0.0000       0.1052         -0.2062         0.2062
  150          0.0000       0.1052         -0.2062         0.2062
  151          0.0000       0.1052         -0.2062         0.2062
  152          0.0000       0.1052         -0.2062         0.2062
  153          0.0000       0.1052         -0.2062         0.2062
  154          0.0000       0.1052         -0.2062         0.2062
  155          0.0000       0.1052         -0.2062         0.2062
  156          0.0000       0.1052         -0.2062         0.2062
```

We observe that: no matter BACK=12 or BACK=0, forecasted future values of Y_t fall into the their 95% confidence intervals that eventually become constant. This means that the final model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}*** is good and involved with the stationary process.

7.2. Forecast Z_t

Let $V_t = log\{Z_t\}$. Then, the final adequate model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}*** can be written as:

$$(1-B)\ (1-B^{12})\ V_t = (1-0.34986*B)\ (1-0.43353*B^{12})a_t$$

where $\hat{\sigma}_a^2$=0.008304.

For a given forecast origin, say *t=132*, forecast can be calculated directly from the difference equation:

$$V_t - V_{t-1} - V_{t-12} + V_{t-13} \cong a_t - 0.35a_{t-1} - 0.43a_{t-12} + 0.15a_{t-13}$$

That is:

$$V_t \cong V_{t-1} + V_{t-12} - V_{t-13} + a_t - 0.35a_{t-1} - 0.43a_{t-12} + 0.15a_{t-13}$$

Then,

$$V_{t+l} \cong V_{t+l-1} + V_{t+l-12} - V_{t+l-13} + a_{t+l} - 0.35a_{t+l-1} - 0.43a_{t+l-12} + 0.15a_{t+l-13}$$

Thus, the *l-step* ahead forecast from the time origin t=132 is given by:

$$\hat{V}_{132}(l) \cong \hat{V}_{132}(l-1) + \hat{V}_{132}(l-12) - \hat{V}_{132}(l-13) + E(a_{132+l}\,|V_{132}, V_{131}, \ldots) - 0.35\,E(a_{132+l-1}\,|V_{132}, V_{131}, \ldots)$$
$$- 0.43\,E(a_{132+l-12}\,|V_{132}, V_{131}, \ldots) + 0.15\,E(a_{132+l-13}\,|V_{132}, V_{131}, \ldots)$$

where

$$\hat{V}_{132}(j) = V_{t+l}, \qquad j \le 0$$

and

$$E(\hat{a}_{132+l}\,|V_{132}, V_{131}, \ldots) = \begin{cases} \hat{a}_{132+j}, & j \le 0 \\ 0, & j > 0 \end{cases}$$

To derive the forecast variance, we first rewrite the model in the following AR representation because the model is non-stationary but invertible:

$$\pi(B)V_t = a_t$$

where

$$\pi(B) = (1 - \pi_1 B - \pi_2 B^2 - \ldots) = \frac{(1-B)(1-B^{12})}{(1-0.35B)(1-0.43B^{12})}.$$

Hence

$$(1 - \pi_1 B - \pi_2 B^2 - \ldots)(1 - 0.35B - 0.43B^{12} + 0.15B^{13}) = (1 - B)(1 - B^{12}).$$

By equating the coefficients of B^j on both sides, we have

$\pi_j = (0.35)^{j-1}(1-0.35) = (0.35)^{j-1}(0.65), \qquad 1 \le j \le 11$

$\pi_{12} = (0.35)^{11}(0.65) - 0.43 + 1 = (0.35)^{11}(0.65) + 0.57$

$\pi_{13} = (0.35)\,\pi_{12} + 0.43\pi_1 + 0.15 - 1 = (0.35)\,\pi_{12} + 0.43\pi_1 - 0.85 = (0.35)\,\pi_{12} - 0.57$

$\pi_j = (0.35)\,\pi_{i-1} + 0.43\pi_{j-12} - 0.15\pi_{j-13} \qquad j \ge 14$

From (5.2.26 on P92), the Ψ_j weights that are needed for calculating the forecast variance can be obtained as below:

$\Psi_1 = \pi_1 = 0.65$

$\Psi_2 = \pi_2 + \pi_1 \Psi_1 = (0.35)(0.65) + (0.65)^2 = 0.65$

.

.

.

$\Psi_j = \sum_{i=0}^{j-1} \pi_{j-i}\psi_i \qquad j = 1, \ldots, l-1.$

The variance of the forecast error is

$$Var[e_{132}(l)] = 0.008304 \sum_{j=0}^{l-1} \psi_j^2$$

and the 95% forecast limits, by (5.2.10 on P90) are given by

$$\hat{V}_{132}(l) \pm 1.96\left(1 + \sum_{j=1}^{l-1} \psi_j^2\right)(0.008304)^{1/2}.$$

Now, we can use the SAS to conduct the above forecast procedure:

- In IDENTIFY statement of PROC ARIMA, VAR=vt(1,12) is the same as ***(1-B) (1-B^{12}) V$_t$*** where **V$_t$= *log{Z$_t$}*** and ***(1-B) (1-B^{12}) V$_t$=Y$_t$***.
- Thus, for the model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z$_t$}*** in this case,

just change *identify var=yt nlag=48;* to *identify var=vt(1,12) nlag=60;*, then we can obtain the forecasts of $\hat{V}_{132}(l)$.

```
options ps=80 ls=78;    /* sets page length and column width */
data data1;             /* name of data set */
  input zt @@;          /* read horizontal data and store in variable zt */
  vt=log(zt);           /* transformation with log */
  wt=dif(vt);           /* (1-B) differencing */
  yt=dif12(wt);         /* (1-B)(1-B12) differening for log(zt) */
  date=_n_;             /* use the case number as date */
  datalines;
   14   18   23   36   39   39   40   40   40   39
   28   13   18   21   23   37   42   42   42   44
   45   43   33   14   15   17   26   33   54   53
   51   54   54   52   37   18   21   26   34   49
   64   65   63   61   60   57   42   23   23   29
   35   47   57   57   56   55   53   52   39   25
   26   27   36   53   63   65   65   64   64   62
   48   27   26   26   35   57   61   62   52   62
   62   60   52   33   32   31   43   71   79   80
   79   75   73   72   59   34   34   32   48   77
   92   94   95   97   97   93   83   43   35   43
   63   97  133  134  133  134  134  133  121   58
   58   51   80  127  150  151  150  151  151  148
  132   63
;
proc gplot data=data1;      /* use gplot of scatter plot */
  plot yt*date;             /* plot ytt versus date */
  symbol i=join v=dot;      /* connect points in plot */
  run;
proc arima data=data1;                /* use arima procedure */
  identify var=vt(1,12) nlag=60;      /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
  forecast lead=24 back=0 out=data2;
run;
quit;
```

We get the first 24 forecasts, i.e., $\hat{V}_{132}(l)$, for $l=1, 2, \ldots, 24$, and their 95% limits, as follows:

```
                       Forecasts for variable vt
 Obs          Forecast     Std Error        95% Confidence     Limits
 133           4.1124       0.0911           3.9338           4.2910
 134           4.0840       0.1087           3.8709           4.2970
 135           4.5010       0.1238           4.2584           4.7436
 136           4.9575       0.1372           4.6885           5.2265
 137           5.1576       0.1495           4.8646           5.4506
 138           5.1667       0.1608           4.8516           5.4818
 139           5.1580       0.1714           4.8221           5.4938
 140           5.1667       0.1813           4.8114           5.5221
 141           5.1653       0.1907           4.7915           5.5392
 142           5.1459       0.1997           4.7544           5.5373
 143           5.0295       0.2083           4.6211           5.4378
 144           4.3167       0.2166           3.8922           4.7412
 145           4.2768       0.2433           3.7999           4.7537
 146           4.2483       0.2604           3.7379           4.7587
 147           4.6654       0.2765           4.1236           5.2073
 148           5.1219       0.2916           4.5503           5.6935
 149           5.3220       0.3060           4.7222           5.9218
 150           5.3311       0.3198           4.7043           5.9579
 151           5.3224       0.3330           4.6697           5.9750
 152           5.3311       0.3457           4.6536           6.0086
```

```
     153          5.3297       0.3579         4.6282         6.0312
     154          5.3102       0.3698         4.5855         6.0349
     155          5.1938       0.3812         4.4467         5.9410
     156          4.4811       0.3924         3.7121         5.2501
```

Thus, we obtain the above is for first 24 forecasts, i.e., $\hat{V}_{132}(l)$ for *l=1, 2, …, 24*, starting from the origin *t=132* (months).
Since $\boldsymbol{V_t = \log\{Z_t\}}$, we have: $\hat{Z}_{132}(l) = \exp\{\hat{V}_{132}(l)\}$ for *l=1, 2, …, 24*.

Table 1 Forecasts of Z_t from the origin *t=132*

Lead time *l*	**Forecasts $\hat{Z}_{132}(l)$**	**Standard error**	**95% Confidence limits**	
1	61.09545	1.095406	51.10261	73.04234
2	59.37985	1.114817	47.98675	73.47791
3	90.10922	1.131776	70.69690	114.85188
4	142.23938	1.147098	108.69389	186.13779
5	173.74955	1.161227	129.62497	232.89422
6	175.33479	1.174438	127.93927	240.28816
7	173.81325	1.186914	124.22921	243.18795
8	175.33952	1.198786	122.89921	250.15578
9	175.09301	1.210149	120.47805	254.46594
10	171.71851	1.221075	116.09272	253.99738
11	152.85032	1.231624	101.60914	229.93227
12	74.94219	1.241839	49.01871	114.57528
13	72.01125	1.275474	44.69799	116.01463
14	69.98912	1.297469	42.01114	116.59947
15	106.20884	1.318451	61.77878	182.59211
16	167.65299	1.338593	94.66399	296.91890
17	204.79302	1.358023	112.41441	373.08547
18	206.66150	1.376842	110.42107	386.78283
19	204.86810	1.395128	106.66846	393.47093
20	206.66707	1.412946	104.96164	406.92270
21	206.37651	1.430348	102.32931	416.21763
22	202.39910	1.447378	98.05586	417.77610
23	180.15976	1.464073	85.34157	380.32510
24	88.33196	1.480465	40.93962	190.58643

From Table 1, we observe that:

- The 24 lead forecasts start from *t=132* (months) with their 95% forecasts limits.
- All 24 lead forecasts fall into 95% confidence interval.
- Because the raw process *{Zt}* is non-stationary with seasonal period 12 months, the 95% limits become wider and wider with seasonal period 12 month as the forecast lead time becomes larger.

Using the following SAS code, we can plot *{Zt}* and its 24 lead forecasts starting from *t=132* (months) with their 95% forecasts limits:

```
data test1;
set work.data2;
ozt=exp(vt);
zt=exp(FORECAST);
STDz=exp(STD);
L95z=exp(L95);
U95z=exp(U95);
date=_n_;
run;
```

```
proc print data=test1;
var ozt zt STDz L95z U95z;
run;
data test3;
set test1;
if date LT 133 then do;
   zt=.;
   STDz=.;
   L95z=.;
   U95z=.;
end;
run;
proc gplot data=test3;
legend1 position=(middle left inside) frame mode=protect label=none across=1
value=('Actual observation' 'Forecast' 'Lower limit' 'Upper limit');
plot ozt*date=1
     zt*date=2
     L95z*date=3
     U95z*date=4 /overlay haxis=axis1 vaxis=axis2 legend=legend1;
title 'Time series data';
axis1 label=('Date');
axis2 label=(angle=90 '{Zt} and its 24 forecasts with C.I.');
symbol1 c=black i=join l=1 v=dot;
symbol2 c=red i=join l=1 v=circle;
symbol3 c=green i=join l=20 v=star;
symbol4 c=blue i=join l=20 v=star;
run;
quit;
```

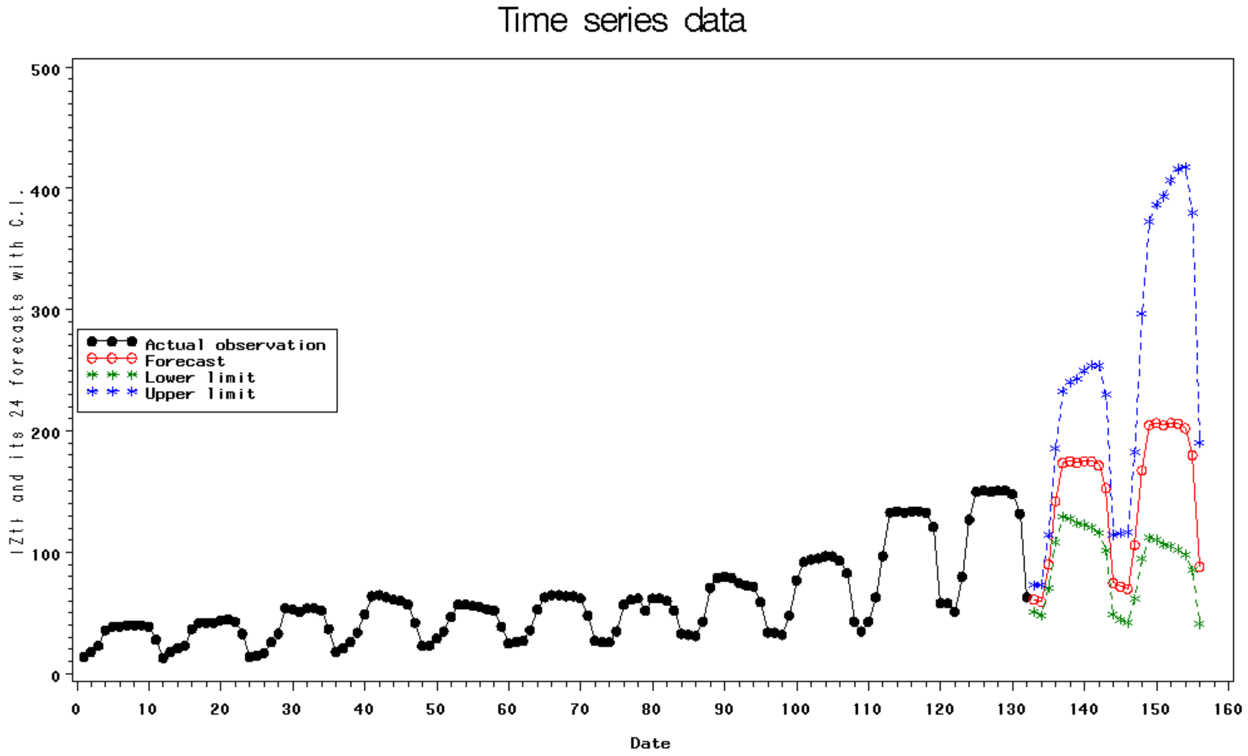

Figure 10 *{Zt}* and its 24 lead forecasts starting from *t=132* with their 95% forecasts limits

As Table 1, Figure 10 also shows that:

- All 24 lead forecasts month by month fall into 95% confidence intervals, and the trend is still upward with increases in variance.

- Because the raw process *{Zt}* is non-stationary with seasonal period 12 months, the 95% limits become wider with seasonal period 12 month as the forecast lead time *l* becomes larger.

NOTE:

If we want to forecast Z_t from the origin *t=120*, just use BACK=12 in the FORECAST statement.

```
proc arima data=data1;                /* use arima procedure */
  identify var=vt(1,12) nlag=60;    /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant printall plot;
  forecast lead=24 back=12 out=data2;
run;
```

Use the similar way like forecasting $\hat{Z}_{132}(l) = exp\{\hat{V}_{132}(l)\}$ starting from t=132, we can also obtain lead forecasts.

8. APPENDIX I: why the final model is selected from METHOD=CLS

Since in the ESTIMATE statement of PROC ARIMA in SAS, there are **three estimation methods** to use:

- METHOD=ML specifies the maximum likelihood method.
- METHOD=ULS specifies the unconditional least-squares method.
- METHOD=CLS specifies the conditional least-squares method.

METHOD=CLS is the default. In other words, all the above estimation analyses in this study are based on **the conditional least-squares method**.

The three estimation methods can lead to the similar conclusion about the form of the final model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z$_t$}***.
However, the three estimation methods give slightly different coefficients θ_1, Θ_1 as in the below:

$$(1-B)(1-B^{12})\log[Z_t] = (1-\theta_1 B)*(1-\Theta_1 B^{12})*a_t$$

Thus, we identify which method yields the better model by checking the values of AIC and SBC and the residuals analysis.

Use the following SAS code:

```
proc arima data=data1;         /*  use arima procedure */
  identify var=yt nlag=60;    /* produce sample acf and pacf */
  estimate q=(1)(12) noconstant method=cls printall plot;
  estimate q=(1)(12) noconstant method=uls printall plot;
  estimate q=(1)(12) noconstant method=ml printall plot;
run;
```

We get:

- For METHOD=CLS

```
                        The ARIMA Procedure
                 Conditional Least Squares Estimation

                                Standard                 Approx
Parameter        Estimate          Error    t Value    Pr > |t|     Lag
MA1,1             0.34986        0.08691       4.03      0.0001       1
MA2,1             0.43353        0.08560       5.06      <.0001      12

                   Variance Estimate      0.008304
                   Std Error Estimate     0.091125
```

```
                      AIC                        -230.444
                      SBC                        -224.886
                      Number of Residuals             119
               * AIC and SBC do not include log determinant.
```

- For METHOD=ULS

```
                        Unconditional Least Squares Estimation
                                   Standard                 Approx
       Parameter      Estimate        Error    t Value    Pr > |t|     Lag
       MA1,1           0.39674      0.08546       4.64      <.0001       1
       MA2,1           0.55139      0.08503       6.48      <.0001      12


                      Variance Estimate       0.007867
                      Std Error Estimate      0.088697
                      AIC                      -232.35
                      SBC                     -226.792
                      Number of Residuals          119
```

- For METHOD=ML

```
                            Maximum Likelihood Estimation
                                   Standard                 Approx
       Parameter      Estimate        Error    t Value    Pr > |t|     Lag
       MA1,1           0.38618      0.08487       4.55      <.0001       1
       MA2,1           0.47595      0.08863       5.37      <.0001      12


                      Variance Estimate       0.007905
                      Std Error Estimate      0.088909
                      AIC                      -233.06
                      SBC                     -227.502
                      Number of Residuals          119
```

Here we observe that: when METHOD=ML, AIC becomes the most negative to *-233.06* and SBC becomes the most negative to *-227.502*. These imply *the maximum likelihood method* can provide improvement over the estimation. Even METHOD=ULS give more negative AIC and SBC than METHOD=CLS.

However, if we check their **ACF** and **PACF** plots, we can know the reason why this study chooses the conditional least-squares method: there is a **spike** at lag 14 for both ACF and PACF generated by METHOD=ULS; There is a **spike** at lag 30 for ACF and a **spike** at lag 14 for PACF generated by METHOD=ML; there is no spike generated by <u>METHOD=CLS</u>.

- For METHOD=CLS

```
                      Autocorrelation Plot of Residuals
Lag  Covariance   Correlation  -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
  0   0.0083038       1.00000  |                    |********************|
  1  0.00009288       0.01119  |                .   |   .                |
  2  -0.0001558       -.01876  |                .   |   .                |
  3  0.00010415       0.01254  |                .   |   .                |
  4  -0.0007902       -.09516  |                . **|   .                |
  5  -0.0010535       -.12687  |                .***|   .                |
  6  -0.0001567       -.01887  |                .   |   .                |
  7  -0.0002055       -.02474  |                .   |   .                |
  8  -0.0002735       -.03294  |                .  *|   .                |
  9  -0.0002564       -.03087  |                .  *|   .                |
 10  0.00014323       0.01725  |                .   |   .                |
 11  0.00063677       0.07668  |                .   |** .                |
 12  -0.0000721       -.00868  |                .   |   .                |
```

```
13  0.00022088      0.02660  |                .   |*  .                |
14  -0.0012106      -.14579  |                .***|   .                |
15  -0.0012667      -.15255  |                .***|   .                |
16  0.00068110      0.08202  |                .   |** .                |
17  -0.0000918      -.01105  |                .   |   .                |
18  -0.0002620      -.03156  |                .  *|   .                |
19  0.00056078      0.06753  |                .   |*  .                |
20  0.00022719      0.02736  |                .   |*  .                |
21  0.00013887      0.01672  |                .   |   .                |
22  -0.0006722      -.08095  |                . **|   .                |
23  0.00013426      0.01617  |                .   |   .                |
24  0.00026635      0.03208  |                .   |*  .                |
25  -0.0004397      -.05296  |                .  *|   .                |
26  -0.0004531      -.05456  |                .  *|   .                |
27  0.00042783      0.05152  |                .   |*  .                |
28  0.00060441      0.07279  |                .   |*  .                |
29  -0.0000513      -.00618  |                .   |   .                |
30   0.0014498      0.17459  |                .   |***.                |
                                                     .
                                                     .


                          Partial Autocorrelations
     Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
      1         0.01119   |                .   |   .                |
      2        -0.01889   |                .   |   .                |
      3         0.01297   |                .   |   .                |
      4        -0.09587   |                . **|   .                |
      5        -0.12527   |                .***|   .                |
      6        -0.02164   |                .   |   .                |
      7        -0.02799   |                .  *|   .                |
      8        -0.04090   |                .  *|   .                |
      9        -0.05735   |                .  *|   .                |
     10        -0.00442   |                .   |   .                |
     11         0.06680   |                .   |*  .                |
     12        -0.02228   |                .   |   .                |
     13         0.01108   |                .   |   .                |
     14        -0.16502   |                .***|   .                |
     15        -0.14903   |                .***|   .                |
     16         0.08918   |                .   |** .                |
     17        -0.01440   |                .   |   .                |
     18        -0.05474   |                .  *|   .                |
     19        -0.00009   |                .   |   .                |
     20         0.00427   |                .   |   .                |
     21         0.02875   |                .   |*  .                |
     22        -0.12256   |                . **|   .                |
     23        -0.01402   |                .   |   .                |
     24         0.02942   |                .   |*  .                |
     25        -0.01828   |                .   |   .                |
     26        -0.05876   |                .  *|   .                |
     27         0.01091   |                .   |   .                |
     28         0.07621   |                .   |** .                |
     29        -0.04989   |                .  *|   .                |
     30         0.16554   |                .   |***.                |.
                                                     .
                                                     .
```

- For METHOD=ULS

```
                    Autocorrelation Plot of Residuals
Lag  Covariance  Correlation  -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
 0   0.0078672      1.00000  |                    |********************|
 1  0.00037827      0.04808  |                .   |*  .                |
 2  -0.0002150      -.02733  |                .  *|   .                |
 3  -0.0000383      -.00487  |                .   |   .                |
 4  -0.0005681      -.07221  |                .  *|   .                |
 5  -0.0009579      -.12175  |                . **|   .                |
```

```
 6  -0.0002393       -.03042 |                .  *|   .                |
 7  -0.0002353       -.02991 |                .  *|   .                |
 8  -0.0002211       -.02811 |                .  *|   .                |
 9  -0.0004013       -.05100 |                .  *|   .                |
10  -0.0000699       -.00888 |                .   |   .                |
11  0.00062259       0.07914 |                .   |** .                |
12  0.00064021       0.08138 |                .   |** .                |
13  0.00009802       0.01246 |                .   |   .                |
14  -0.0013950       -.17732 |                ****|   .                |
15  -0.0012419       -.15786 |                .***|   .                |
16  0.00054808       0.06967 |                .   |*  .                |
17  -0.0001702       -.02163 |                .   |   .                |
18  -0.0001833       -.02330 |                .   |   .                |
19  0.00051089       0.06494 |                .   |*  .                |
20  0.00018848       0.02396 |                .   |   .                |
21  0.00005224       0.00664 |                .   |   .                |
22  -0.0006511       -.08276 |                . **|   .                |
23  0.00032964       0.04190 |                .   |*  .                |
24  0.00049278       0.06264 |                .   |*  .                |
25  -0.0006076       -.07723 |                . **|   .                |
26  -0.0008193       -.10414 |                . **|   .                |
27  0.00021174       0.02691 |                .   |*  .                |
28  0.00057869       0.07356 |                .   |*  .                |
29  -3.1171E-7       -.00004 |                .   |   .                |
30   0.0013677       0.17385 |                .   |***.                |
                                                     .
                                                     .
```

```
                     Partial Autocorrelations
Lag    Correlation    -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
 1        0.04808    |                .   |*  .                |
 2       -0.02971    |                .  *|   .                |
 3       -0.00209    |                .   |   .                |
 4       -0.07291    |                .  *|   .                |
 5       -0.11579    |                . **|   .                |
 6       -0.02454    |                .   |   .                |
 7       -0.03606    |                .  *|   .                |
 8       -0.03388    |                .  *|   .                |
 9       -0.06951    |                .  *|   .                |
10       -0.02601    |                .  *|   .                |
11        0.06604    |                .   |*  .                |
12        0.06321    |                .   |*  .                |
13       -0.00527    |                .   |   .                |
14       -0.19851    |                ****|   .                |
15       -0.15614    |                .***|   .                |
16        0.09615    |                .   |** .                |
17       -0.01275    |                .   |   .                |
18       -0.04730    |                .  *|   .                |
19       -0.00267    |                .   |   .                |
20       -0.00340    |                .   |   .                |
21        0.02527    |                .   |*  .                |
22       -0.12116    |                . **|   .                |
23       -0.00194    |                .   |   .                |
24        0.03960    |                .   |*  .                |
25       -0.05213    |                .  *|   .                |
26       -0.07292    |                .  *|   .                |
27        0.01385    |                .   |   .                |
28        0.05292    |                .   |*  .                |
29       -0.05402    |                .  *|   .                |
30        0.16788    |                .   |***.                |
                                                .
                                                .
```

- For METHOD=ML

```
                 Autocorrelation Plot of Residuals
Lag  Covariance  Correlation  -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
  0  0.0079048      1.00000   |                    |********************|
  1 0.00026811      0.03392   |                .   |*  .                |
  2 -0.0000817      -.01034   |                .   |   .                |
  3 0.00007537      0.00954   |                .   |   .                |
  4 -0.0006419      -.08120   |                . **|   .                |
  5 -0.0009893      -.12515   |                .***|   .                |
  6 -0.0002410      -.03049   |                .  *|   .                |
  7 -0.0002255      -.02853   |                .  *|   .                |
  8 -0.0002118      -.02679   |                .  *|   .                |
  9 -0.0003141      -.03973   |                .  *|   .                |
 10 0.00004902      0.00620   |                .   |   .                |
 11 0.00057633      0.07291   |                .   |*  .                |
 12 0.00010291      0.01302   |                .   |   .                |
 13 0.00010480      0.01326   |                .   |   .                |
 14 -0.0012764      -.16147   |                .***|   .                |
 15 -0.0012529      -.15850   |                .***|   .                |
 16 0.00057308      0.07250   |                .   |*  .                |
 17 -0.0001345      -.01701   |                .   |   .                |
 18 -0.0002494      -.03155   |                .  *|   .                |
 19 0.00051237      0.06482   |                .   |*  .                |
 20 0.00024371      0.03083   |                .   |*  .                |
 21 0.00014278      0.01806   |                .   |   .                |
 22 -0.0006086      -.07699   |                . **|   .                |
 23 0.00021571      0.02729   |                .   |*  .                |
 24 0.00025417      0.03215   |                .   |*  .                |
 25 -0.0005776      -.07307   |                .  *|   .                |
 26 -0.0006160      -.07793   |                . **|   .                |
 27 0.00031169      0.03943   |                .   |*  .                |
 28 0.00055877      0.07069   |                .   |*  .                |
 29 -0.0000165      -.00209   |                .   |   .                |
 30  0.0014016      0.17731   |                .   |****                |
                                                      .
                                                      .
```

```
                   Partial Autocorrelations
Lag    Correlation   -1 9 8 7 6 5 4 3 2 1 0 1 2 3 4 5 6 7 8 9 1
 1        0.03392    |                .   |*  .                |
 2       -0.01150    |                .   |   .                |
 3        0.01029    |                .   |   .                |
 4       -0.08212    |                . **|   .                |
 5       -0.12022    |                . **|   .                |
 6       -0.02563    |                .  *|   .                |
 7       -0.02883    |                .  *|   .                |
 8       -0.03067    |                .  *|   .                |
 9       -0.05974    |                .  *|   .                |
10       -0.01206    |                .   |   .                |
11        0.06198    |                .   |*  .                |
12       -0.00157    |                .   |   .                |
13       -0.00160    |                .   |   .                |
14       -0.18329    |                ****|   .                |
15       -0.15565    |                .***|   .                |
16        0.09068    |                .   |** .                |
17       -0.01615    |                .   |   .                |
18       -0.05748    |                .  *|   .                |
19       -0.00418    |                .   |   .                |
20        0.00102    |                .   |   .                |
21        0.02600    |                .   |*  .                |
22       -0.11840    |                . **|   .                |
23       -0.00752    |                .   |   .                |
24        0.02145    |                .   |   .                |
25       -0.04420    |                .  *|   .                |
26       -0.07279    |                .  *|   .                |
27        0.00599    |                .   |   .                |
```

```
      28        0.06062     |                .   |*  .              |
      29       -0.05803     |                .  *|   .              |
      30        0.16423     |                .   |***.              |
                                                        .
```

The **spikes** for both METHOD=ML and METHOD=ULS may cause some small effect from non-stationarity of residuals. Therefore, we choose <u>METHOD=CLS with stationary residuals that has no spike</u>.

9. APPENDIX II: forecast using the final model selected from METHOD=ML

According to Section 8, <u>we can still chose the final model selected from METHOD=ML </u>since it produces nice AIC and SBC, and the spike at lag 30 for ACF and the spike at lag 14 for PACF are small.
More importantly, portmanteau test for un-correlation among $\hat{a}_t$'s showing below indicates: all values of χ^2 are not big enough and p-value's are larger than 0.05. Then the H_0 of uncorrelated errors is not rejected.

```
                      Autocorrelation Check of Residuals
  To      Chi-          Pr >
 Lag    Square    DF   ChiSq   --------------Autocorrelations--------------

   6      3.09     4  0.5434    0.034  -0.010   0.010  -0.081  -0.125  -0.030
  12      4.23    10  0.9365   -0.029  -0.027  -0.040   0.006   0.073   0.013
  18     12.22    16  0.7285    0.013  -0.161  -0.158   0.072  -0.017  -0.032
  24     14.16    22  0.8956    0.065   0.031   0.018  -0.077   0.027   0.032
  30     22.04    28  0.7794   -0.073  -0.078   0.039   0.071  -0.002   0.177
  36     24.43    34  0.8866    0.057  -0.078  -0.060  -0.006  -0.029  -0.022
  42     25.88    40  0.9589    0.038  -0.057  -0.039   0.027   0.001  -0.035
  48     34.46    46  0.8945    0.087  -0.053  -0.133   0.002   0.115  -0.050
  54     42.61    52  0.8201   -0.021  -0.085   0.052  -0.008   0.139   0.089
  60     44.74    58  0.8992    0.055  -0.070  -0.020  -0.018  -0.018  -0.016
```

The final adequate model **ARIMA(0,1,1)×(0,1,1)$_{12}$ without constant for *log{Z_t}*** using t*he maximum likelihood method* can be written as:

$$(1-B)\,(1-B^{12})\,\log\{Z_t\} = (1-0.386186*B)\,(1-0.47595*B^{12})a_t$$

where $\hat{\sigma}_a^2$=0.007905.

Use the following SAS code:

```
options ps=80 ls=78;    /* sets page length and column width */
data data1;             /* name of data set */
  input zt @@;          /* read horizontal data and store in variable zt */
  vt=log(zt);           /* transformation with log */
  wt=dif(vt);           /* (1-B) differencing */
  yt=dif12(wt);         /* (1-B)(1-B12) differening for log(zt) */
  date=_n_;             /* use the case number as date */
  datalines;
   14    18    23    36    39    39    40    40    40    39
   28    13    18    21    23    37    42    42    42    44
   45    43    33    14    15    17    26    33    54    53
   51    54    54    52    37    18    21    26    34    49
   64    65    63    61    60    57    42    23    23    29
   35    47    57    57    56    55    53    52    39    25
   26    27    36    53    63    65    65    64    64    62
   48    27    26    26    35    57    61    62    52    62
   62    60    52    33    32    31    43    71    79    80
```

```
   79   75   73   72   59   34   34   32   48   77
   92   94   95   97   97   93   83   43   35   43
   63   97   133  134  133  134  134  133  121  58
   58   51   80   127  150  151  150  151  151  148
   132  63
;
proc gplot data=data1;      /* use gplot of scatter plot */
  plot yt*date;             /* plot ytt versus date */
  symbol i=join v=dot;      /* connect points in plot */
  run;
proc arima data=data1;        /* use arima procedure */
  identify var=yt nlag=60;    /* produce sample acf and pacf for yt=(1-B)(1-B12)vt */
  estimate q=(1)(12) noconstant printall plot;
  identify var=vt(1,12) nlag=60;      /* produce sample acf and pacf for vt */
  estimate q=(1)(12) noconstant method=ml printall plot;      /* ML method */
  forecast lead=24 back=0 out=data2;
run;
quit;
data test1;
set work.data2;
ozt=exp(vt);
zt=exp(FORECAST);
STDz=exp(STD);
L95z=exp(L95);
U95z=exp(U95);
date=_n_;
run;
proc print data=test1;
var ozt zt STDz L95z U95z;
run;

data test3;
set test1;
if date LT 133 then do;
   zt=.;
   STDz=.;
   L95z=.;
   U95z=.;
end;
run;
proc gplot data=test3;
legend1 position=(middle left inside) frame mode=protect label=none across=1
value=('Actual observation' 'Forecast' 'Lower limit' 'Upper limit');
plot ozt*date=1
     zt*date=2
     L95z*date=3
     U95z*date=4 /overlay haxis=axis1 vaxis=axis2 legend=legend1;
title 'Time series data';
axis1 label=('Date');
axis2 label=(angle=90 '{Zt} and its 24 forecasts with C.I.');
symbol1 c=black i=join l=1 v=dot;
symbol2 c=red i=join l=1 v=circle;
symbol3 c=green i=join l=20 v=star;
symbol4 c=blue i=join l=20 v=star;
run;
quit;
```

We get the first 24 forecasts, i.e., $\hat{V}_{132}(l)$, for *l=1, 2, …, 24*, and their 95% limits, as follows:

```
                          Time series data
             Obs     ozt       zt       STDz       L95z       U95z
             133      .      61.318   1.09298    51.512     72.991
```

```
134       .     59.976    1.10996     48.886     73.583
135       .     90.572    1.12495     71.908    114.081
136       .    142.915    1.13856    110.820    184.305
137       .    174.545    1.15117    132.458    230.005
138       .    176.206    1.16298    131.069    236.886
139       .    174.491    1.17415    127.385    239.018
140       .    176.127    1.18479    126.325    245.563
141       .    175.813    1.19498    124.001    249.275
142       .    172.363    1.20479    119.636    248.330
143       .    153.042    1.21426    104.607    223.902
144       .     75.521    1.22343     50.864    112.129
145       .     72.612    1.25309     46.662    112.992
146       .     71.023    1.27183     44.332    113.782
147       .    107.254    1.28973     65.139    176.599
148       .    169.237    1.30693    100.148    285.988
149       .    206.692    1.32354    119.323    358.033
150       .    208.659    1.33962    117.640    370.099
151      .     206.629    1.35525    113.877    374.926
152      .     208.566    1.37048    112.455    386.819
153      .     208.194    1.38535    109.905    394.382
154      .     204.109    1.39989    105.565    394.641
155      .     181.229    1.41414     91.889    357.429
156      .      89.430    1.42813     44.478    179.813
```

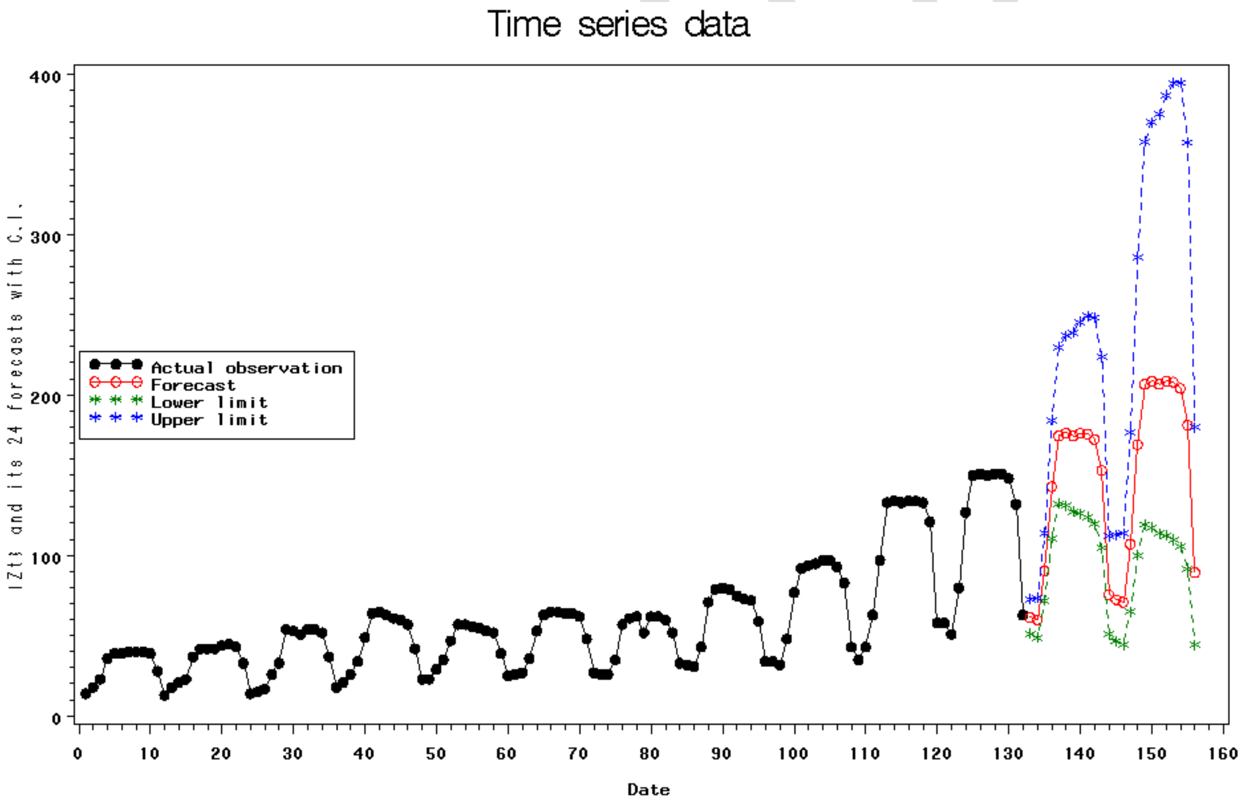

Figure 11 *{Zt}* and its 24 lead forecasts starting from *t=132* with their 95% forecasts limits using METHOD=ML

We notice that:

- METHOD=ML *the maximum likelihood method* produces the very similar forecasts as METHOD=CLS *the conditional least-squares method* does for the first 24 leads starting from *t=132*.
- METHOD=ML produces narrower 95% confidence intervals than METHOD=CLS does.
- METHOD=ML can also produce reasonable forecasts.